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SPECTRAL AND FREDHOLM PROPERTIES OF OPERATORS IN ELEMENTARY NEST ALGEBRAS

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ABSTRACT. Some spectral and Fredholm properties are proved for linear operators which leave invariant certain nests of closed subspaces.

1. INTRODUCTION

Throughout, X is an infinite-dimensional Banach space, $B(X)$ is the algebra of all bounded linear operators on X, and for $T \in B(X)$, $\sigma(T)$ is the spectrum of T. For $T \in B(X)$, $\mathrm{Lat}(T)$ is the collection of all closed subspaces M of X such that $T(M) \subseteq M$. When $\mathrm{Lat}(T)$ contains certain types of nests (totally ordered collections), then this may affect the spectral or Fredholm properties of T. A famous example is due to J. Ringrose: When T is a compact operator and $\mathrm{Lat}(T)$ contains a continuous nest, then $\sigma(T) = \{0\}$ [3, Corollary 4.3.11].

In this paper we consider the spectral and Fredholm theory of operators T such that $\mathrm{Lat}(T)$ contains an elementary nest of one of the two types described below.

Definition 1. A collection $\mathscr{E}$ of closed subspaces of X is an *elementary upper nest* (EUN) if $\mathscr{E} = \{M_0 : 0 \leq n \leq \infty\}$, where

(i) $M_0 = \{0\}$, $M_\infty = X$, and $M_n \subseteq M_{n+1}$ for $n \geq 0$;
(ii) M_n is f.d. (finite-dimensional) for $0 \leq n < \infty$;
(iii) $\bigcup\{M_n : 0 \leq n < \infty\}$ is dense in X.

The collection $\mathscr{E}$ as above is an *elementary lower nest* (ELN) if

(i) $M_\infty = \{0\}$, $M_0 = X$, and $M_{n+1} \subseteq M_n$ for $n \geq 0$;
(ii) M_n/M_{n+1} is f.d. for $n \geq 0$;
(iii) $\bigcap\{M_n : 0 \leq n < \infty\} = \{0\}$.

As an example, assume $\{X_k : k \geq 1\}$ is a linearly independent collection of f.d. subspaces of X such that $X = \bigoplus \sum_{k=1}^{\infty} X_k$ (for each $x \in X$, there exists a unique sequence $\{x_k\}_{k\geq 1}$ with $x_k \in X_k$, $k \geq 1$, and $x = \sum_{k=1}^{\infty} x_k$). Setting $M_0 = \{0\}$, $M_\infty = X$, and $M_n = \sum_{k=1}^{n} X_k$, we have $\mathscr{E} = \{M_n : 0 \leq n \leq \infty\}$ is an EUN. An operator $S \in B(X)$ such that $S(X_n) \subseteq X_{n-1}$ for $n \geq 1$ $(X_0 = \{0\})$ is a type of backward shift operator. Clearly, $\mathscr{E} \subseteq \mathrm{Lat}(S)$. Results in this paper

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show that $\sigma(S)$ is connected [Theorem 6], and that if $\lambda - S$ is Fredholm on X, then $\operatorname{ind}(\lambda - S) \leq 0$ [Corollary 8]. Here $\operatorname{ind}(T)$ denotes the usual index of a Fredholm operator $T \in B(X)$.

Also, note in this same situation that if $N_0 = X$, $N_\infty = \{0\}$, and $N_n = \{x = \sum_{k=1}^\infty x_k \in X : x_k = 0,\ 1 \leq k \leq n\}$, $1 \leq n < \infty$, then $\mathscr{N} = \{N_n : 0 \leq n \leq \infty\}$ is an ELN.

2. Spectral properties

Throughout T is an operator in $B(X)$. The object of this section is to show that when $\mathscr{E}$ is an EUN or an ELN, and $\mathscr{E} \subseteq \operatorname{Lat}(T)$, then this affects the spectral properties of T. Let $\operatorname{Alg}(\mathscr{E}) = \{S \in B(X) : \mathscr{E} \subseteq \operatorname{Lat}(S)\}$. The algebra $\operatorname{Alg}(\mathscr{E})$ is a closed subalgebra of $B(X)$ which contains the identity operator. Properties of compact operators in $\operatorname{Alg}(\mathscr{N})$, where $\mathscr{N}$ is an elementary nest, are studied in [2].

Proposition 2. *When $\mathscr{E}$ is either an EUN or an ELN, then $A = \operatorname{Alg}(\mathscr{E})$ is inverse closed in $B(X)$ (that is, if $T \in A$ and $T^{-1} \in B(X)$, then $T^{-1} \in A$).*

Proof. Assume $\mathscr{E} = \{M_n : 0 \leq n \leq \infty\}$. It suffices in either case to show that if $T \in A$ and $T^{-1} \in B(X)$, then $T(M_n) = M_n$ for all n. When $\mathscr{E}$ is an EUN, then this is obvious since T is 1-1 on M_n and M_n is f.d., so $T(M_n) = M_n$. Now suppose $\mathscr{E}$ is an ELN. We do this case by induction. Certainly $T(M_0) = M_0$. Suppose $T(M_n) = M_n$ for some n. This implies T maps M_n/M_{n+1} onto M_n/M_{n+1}. Therefore T is 1-1 on M_n/M_{n+1}. Fix $z \in M_{n+1}$. By the induction hypothesis $\exists w \in M_n$ such that $Tw = z$. Thus, $T(w + M_{n+1}) = z + M_{n+1}$. Since T is 1-1 on M_n/M_{n+1}, $w \in M_{n+1}$.

Next we prove a key result. For $T \in B(X)$, let $\mathscr{N}(T)$ be the null space of T and $\mathscr{R}(T)$ be the range of T.

Theorem 3. *Assume $\mathscr{E} \subseteq \operatorname{Lat}(T)$.*

(1) *If $\mathscr{E}$ is EUN and T is surjective on M_{n+1}/M_n for $n \geq 0$, then $\mathscr{R}(T)$ is dense in X.*

(2) *If $\mathscr{E}$ is an ELN and T is injective on M_n/M_{n+1} for $n \geq 0$, then $\mathscr{N}(T) = \{0\}$.*

(3) *If $\mathscr{E}$ is an ELN and $\mathscr{N}(T) \neq \{0\}$, then $\mathscr{R}(T)$ is not dense in X.*

Proof. Assume T is as in (1). Since T is surjective on M_1/M_0, $M_0 = \{0\}$, then $T(M_1) = M_1$. We verify by induction that $M_n = T(M_n)$ for all $n \geq 1$. Assume $T(M_n) = M_n$. By hypothesis $T(M_{n+1}/M_n) = M_{n+1}/M_n$. Assume $z \in M_{n+1}$. Then $\exists w \in M_{n+1}$ such that $z - T(w) \in M_n$. By the induction hypothesis, $\exists x \in M_n$ with $T(x) = z - T(w)$. Thus $x + w \in M_{n+1}$ and $T(x+w) = z$. Therefore, $T(M_{n+1}) = M_{n+1}$. It follows that $\bigcup_{n=1}^\infty M_n \subseteq \mathscr{R}(T)$, so $\mathscr{R}(T)$ is dense in X.

Now assume T is as in (2). Suppose $y \in X$, $y \neq 0$, and $T(y) = 0$. Choose n such that $y \in M_n$ and $y \notin M_{n+1}$. Then $y + M_{n+1} \in M_n/M_{n+1}$, $y + M_{n+1} \neq 0 + M_{n+1}$. Also $T(y + M_{n+1}) = T(y) + M_{n+1} = 0 + M_{n+1}$, a contradiction.

Assume that T is as in (3), so $\mathscr{N}(T) \neq \{0\}$. By (2), $\exists n$ such that T is not 1-1 on M_n/M_{n+1}. Since M_n/M_{n+1} is f.d., $T(M_n/M_{n+1})$ is a proper subspace of M_n/M_{n+1}. This implies $\exists Z$ an f.d. subspace such that $T(M_n) \subseteq Z \oplus M_{n+1} \neq M_n$. Now $X = Y \oplus M_n$, where Y is an f.d. subspace. $T(X) \subseteq T(Y) + T(M_n) \subseteq$

$T(Y) + Z + M_{n+1}$. This last subspace is a closed proper subspace of X, so (3) holds.

We use the following notation for various parts of the spectrum of an operator $T \in B(X)$:

$$\begin{aligned}\sigma_p(T) &\equiv \text{ the point spectrum (eigenvalues) of } T;\\ \sigma_c(T) &\equiv \text{ the continuous spectrum of } T\\ &\equiv \{\lambda \notin \sigma_p(T) : \overline{\mathscr{R}(\lambda - T)} = X, \mathscr{R}(\lambda - T) \neq X\};\\ \sigma_r(T) &\equiv \text{ the residual spectrum of } T \equiv \{\lambda \notin \sigma_p(T) : \overline{\mathscr{R}(\lambda - T)} \neq X\}.\end{aligned}$$

Corollary 4. *Assume* $\mathscr{E} \subseteq \mathrm{Lat}(T)$.

(1) *If* $\mathscr{E}$ *is an EUN, then* $\sigma(T) = \sigma_p(T) \cup \sigma_c(T)$.

(2) *If* $\mathscr{E}$ *is an ELN, and* $T(M_n) \subseteq M_{n+1}$ *for* $n \geq 0$, *then* $\sigma_p(T)\backslash\{0\}$ *is empty.*

Proof. (1) It suffices to show that $\sigma_r(T)$ is empty. Suppose not, so $\exists\lambda$ with $\mathscr{N}(\lambda - T) = \{0\}$ and $\mathscr{R}(\lambda - T)$ not dense in X. By Theorem 3(1), $\lambda - T$ is not surjective on M_{n+1}/M_n for some n. This implies that $\lambda - T$ is not surjective on the f.d. space M_{n+1}. Therefore, $\lambda - T$ is not 1-1 on M_{n+1}. Thus, $\mathscr{N}(\lambda - T) \neq \{0\}$, a contradiction.

(2) Assume $\lambda \neq 0$, $(\lambda - T)x = 0$, and $x \neq 0$. Since $\bigcap\{M_k : 0 \leq k < \infty\} = \{0\}$, $\exists n$ such that $x \notin M_{n+1}$. We may assume n is the smallest nonnegative integer with this property, so $x \in M_n$ and $x \notin M_{n+1}$. But $\lambda x = Tx \in M_{n+1}$ by hypothesis. Thus $x \in M_{n+1}$, a contradiction.

Assume J is a proper closed ideal in a Banach algebra A with identity. It is a fact, from the holomorphic operational calculus, that when $T \in J$ and f is holomorphic on some open neighborhood of $\sigma(T)$ and $f(0) = 0$, then $f(T) \in J$. We apply this to the situation when $T \in J$ has disconnected spectrum.

Assertion 5. *When* $T \in J$ *has disconnected spectrum in* A, *then some nonzero spectral idempotent of* T *is in* J.

To verify this, suppose $\sigma_A(T) = \Delta \cup \Gamma$, where Δ and Γ are both nonempty open and closed subsets of $\sigma_A(T)$ and Δ and Γ are disjoint. Assume $0 \notin \Delta$. Choose disjoint open subsets U and V of the complex plane such that $\Delta \subseteq U$ and $\Gamma \cup \{0\} \subseteq V$. Let $f \equiv 1$ on U, $f \equiv 0$ on V. Then, as remarked above, $F(T) \in J$. Thus Assertion 5 holds.

Theorem 6. *Assume* $\mathscr{E} = \{M_k : 0 \leq k \leq \infty\} \subseteq \mathrm{Lat}(T)$.

(1) *If* $\mathscr{E}$ *is an EUN and* $T(M_n) \subseteq M_{n-1}$ *for* $1 \leq n < \infty$, *then* $\sigma(T)$ *is connected.*

(2) *If* $\mathscr{E}$ *is an ELN and* $T(M_{n-1}) \subseteq M_n$ *for* $1 \leq n < \infty$, *then* $\sigma(T)$ *is connected.*

Proof. Let $A = \mathrm{Alg}(\mathscr{E})$. When $\mathscr{E}$ and T are as in (1), let

$$J = \{S \in A : S(M_n) \subseteq M_{n-1} \text{ for } 1 \leq n < \infty\}. \tag{3}$$

When $\mathscr{E}$ and T are as in (2), let

$$J = \{S \in A : S(M_{n-1}) \subseteq M_n \text{ for } 1 \leq n < \infty\}. \tag{4}$$

In both cases J is a proper closed ideal of A and $T \in J$. We claim that in both cases J contains no nonzero projection. Once this fact is established, the theorem follows from Assertion 5 above.

Assume $E = E^2 \in J$. Let J be as in (3), and $\mathscr{E}$ an EUN. If $E \neq 0$, then $E(M_n) \neq \{0\}$ for some n. Therefore $\exists x \in M_n$, $x \neq 0$, with $x = Ex$. Then $x = Ex \in M_{n-1}$ since $E \in J$. Repeating this argument n times, we have $x = Ex \in M_0 = \{0\}$, a contradiction.

Now assume J is as in (4), and $\mathscr{E}$ is an ELN. If $E \neq 0$, $\exists x \in M_0 = X$, $x \neq 0$, with $x = Ex$. Suppose $x \in M_n$. Then $x = Ex \in M_{n+1}$. Thus by induction it follows that $x \in M_n$ for all n. Therefore $x \in \bigcap\{M_n : 0 \leq n < \infty\} = \{0\}$, a contradiction.

Theorem 6 applies to shift-type and backward shift-type operators. In fact, the spectrum of these operators is often a disk centered at 0. We illustrate this with an example. Assume that $X = \bigoplus \sum_{k=1}^{\infty} X_k$, where each X_k is f.d. (as in the Introduction). For each $\mathscr{O} \in \mathbb{R}$, assume that the operator $V_{\mathscr{O}}$ defined below is everywhere defined and bounded on X: For $x \in X$, $x = \sum_{k=1}^{\infty} x_k$, $x_k \in X_k$,

$$V_{\mathscr{O}}(x) = \sum_{k=1}^{\infty} e^{ik\mathscr{O}} x_k .$$

Then clearly $V_{(-\mathscr{O})} = V_{\mathscr{O}}^{-1}$ for all $\mathscr{O} \in \mathbb{R}$. Now assume $S \in B(X)$ and $S(X_n) \subseteq X_{n-1}$ for $n \geq 1$. (S is a backward shift-type operator.) Fix $x \in X$, $x = \sum_{k=1}^{\infty} x_k$, $x_k \in X_k$, and let $y_{k-1} = S(x_k) \in X_{k-1}$ (as before, $y_0 = 0$). Then

$$V_{\mathscr{O}}^{-1} S V_{\mathscr{O}} x = V_{\mathscr{O}}^{-1} S \sum_{k=1}^{\infty} e^{ik\mathscr{O}} x_k = V_{\mathscr{O}}^{-1} \sum_{k=2}^{\infty} e^{ik\mathscr{O}} y_{k-1} = e^{i\mathscr{O}} \left(\sum_{k=2}^{\infty} y_{k-1} \right) = e^{i\mathscr{O}} S x .$$

Thus, $e^{i\mathscr{O}} S$ is similar to S, and therefore

$$\sigma(S) = e^{i\mathscr{O}} \sigma(S) \quad \text{for all } \mathscr{O} \in \mathbb{R} .$$

This means that when $\lambda \in \sigma(S)$, the circle $\{e^{i\mathscr{O}}\lambda : \mathscr{O} \in \mathbb{R}\} \subseteq \sigma(S)$. Since $0 \in \sigma(S)$ and $\sigma(S)$ is connected [Theorem 6], it follows that $\sigma(S)$ is a disk centered at 0. Results of this type are well known for certain shifts and backward shifts; see [5] for example.

3. Fredholm properties

In this section we consider the Fredholm properties of an operator T when $\mathscr{E}$ is an EUN or an ELN and $\mathscr{E} \subseteq \mathrm{Lat}(T)$. First we establish a basic perturbation result.

When $A = \mathrm{Lat}(\mathscr{E})$, we set $\mathscr{F}(A) \equiv$ the space of all operators in A which have f.d. range.

Proposition 7. *Let $A = \mathrm{Alg}(\mathscr{E})$, and assume $T \in A$.*

(1) *If $\mathscr{E}$ is an EUN, then $\exists K \in \overline{\mathscr{F}(A)}$ such that $\mathscr{R}(T + K)$ is dense in X.*

(2) *If $\mathscr{E}$ is an ELN, then $\exists J \in \overline{\mathscr{F}(A)}$ such that $\mathscr{N}(T + J) = \{0\}$.*

Proof. First assume that $\mathscr{E}$ is an EUN. For each $n \geq 1$ choose an f.d. subspace Y_n of X such that $M_n = Y_n \oplus M_{n-1}$. Let E_n be a projection in $B(X)$ such that $E_n(M_{n-1}) = \{0\}$ and $\mathscr{R}(E_n) = Y_n$. Then

$$E_n(M_j) = \{0\} \text{ for } 1 \leq j \leq n-1, \quad \text{and} \quad E_n(M_j) = Y_n \subseteq M_j \text{ for } n \leq j < \infty .$$

It follows that $E_n \in \mathscr{F}(A)$. Now E_n acts as the identity operator on M_n/M_{n-1}, and the spectrum of T on M_n/M_{n-1} is finite. Therefore, we can choose $\varepsilon_n > 0$ such that $\varepsilon_n\|E_n\| < 2^{-n}$ and $T + \varepsilon_n E_n$ is invertible on M_n/M_{n-1}. Set $K = \sum_{n=1}^{\infty} \varepsilon_n E_n \in \overline{\mathscr{F}(A)}$. By the construction $T+K$ is invertible on M_n/M_{n-1} for $n \geq 1$. It follows from Theorem 3(1) that $\mathscr{R}(T+K)$ is dense in X.

Now assume $\mathscr{E}$ is an ELN. For $1 \leq n < \infty$, choose an f.d. subspace Z_n such that $M_{n-1} = Z_n \oplus M_n$. Let F_n be a projection in $B(X)$ with $F_n(M_n) = \{0\}$ and $\mathscr{R}(F_n) = Z_n$. Then

$$F_n(M_j) \subseteq F_n(M_n) = \{0\} \quad \text{for } n \leq j < \infty,$$

and

$$F_n(M_j) = Z_n \subseteq M_j \quad \text{for } 1 \leq j < n.$$

Therefore, $F_n \in \mathscr{F}(A)$. As before, choose $\varepsilon_n > 0$ with $\varepsilon_n\|F_n\| < 2^{-n}$ and $T+\varepsilon_n F_n$ invertible on M_{n-1}/M_n, $1 \leq n < \infty$. Let $J = \sum_{n=1}^{\infty} \varepsilon_n F_n \in \overline{\mathscr{F}(A)}$. By the construction $T + J$ is invertible on M_{n-1}/M_n for $1 \leq n < \infty$. Therefore, by Theorem 3(2), $\mathscr{N}(T+J) = \{0\}$.

Let $\Phi(X)$ denote the set of Fredholm operators in $B(X)$. Also, let $\Phi^0(X)$ be the set of all $T \in \Phi(X)$ such that $\operatorname{ind}(T) = 0$ ($\operatorname{ind}(T) \equiv$ the index of T).

Corollary 8. *Assume* $\mathscr{E} \subseteq \operatorname{Lat}(T)$ *and* $T \in \Phi(X)$.

(1) *If* $\mathscr{E}$ *is an EUN, then* $\operatorname{ind}(T) \geq 0$.

(2) *If* $\mathscr{E}$ *is an ELN, then* $\operatorname{ind}(T) \leq 0$.

Proof. We prove (1) only (the proof of (2) is similar). Assume $\mathscr{E}$ is an EUN, $\mathscr{E} \subseteq \operatorname{Lat}(T)$, and $T \in \Phi(X)$. By Proposition 7(1), there exists a compact operator K such that $\mathscr{R}(T+K)$ is dense in X. Since $T+K \in \Phi(X)$, we have $\mathscr{R}(T+K) = X$. Thus, $\operatorname{ind}(T) = \operatorname{ind}(T+K) \geq 0$.

Theorem 9. *Assume* $\mathscr{E}$ *is either an EUN or an ELN. Set* $A = \operatorname{Alg}(\mathscr{E})$. *The following are equivalent for* $T \in A$:

(1) $T \in \Phi^0(X)$;

(2) T *is invertible in* A *modulo* $\mathscr{F}(A)$;

(3) $\exists F \in \mathscr{F}(A)$ *such that* $T+F$ *is invertible.*

Proof. We assume $\mathscr{E}$ is an EUN. The proof when $\mathscr{E}$ is an ELN is similar.

Assume $T \in \Phi^0(X)$. By Proposition 7, $\exists K \in \overline{\mathscr{F}(A)}$ such that $\mathscr{R}(T+K)$ is dense. Now $T+K \in \Phi^0(X)$, so $\mathscr{R}(T+K) = X$ and $\mathscr{N}(T+K) = \{0\}$. It follows that $T+K$ is invertible in $B(X)$, therefore in A [Proposition 2]. Since $K \in \overline{\mathscr{F}(A)}$ and the set of invertibles in A is open, $\exists F \in \mathscr{F}(A)$ such that $T+F$ is invertible. This proves (3).

It is clear that (3) implies both (1) and (2). Assume (2) holds, so $\exists S \in A$ and $\exists F, G \in \mathscr{F}(A)$ such that $ST = I - F$ and $TS = I - G$. Now both $S, T \in A$ and $S, T \in \Phi(X)$, so by Corollary 8 $\operatorname{ind}(T) \geq 0$ and $\operatorname{ind}(S) \geq 0$. Also, $0 = \operatorname{ind}(I-F) = \operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$. Therefore, $\operatorname{ind}(T) = 0$, so (1) holds.

For $T \in B(X)$, let $\sigma_W(T)$ denote the Weyl spectrum of T,

$$\sigma_W(T) = \{\lambda \in \mathbf{C} : (\lambda - T) \notin \Phi^0(X)\}.$$

It is well known that

$$\sigma_W(T) = \bigcap \{\sigma(T+K) : K \in B(X),\ K \text{ compact}\},$$

[4, Theorem 5.4, p. 180].

Corollary 10. *Assume $\mathscr{E} \subseteq \mathrm{Lat}(T)$, and $\mathscr{E}$ is either an EUN or an ELN. Then*

$$\sigma_W(T) = \bigcap\{\sigma(T+J) : J \in \mathscr{F}(A)\}.$$

Proof. The inclusion $\sigma_W(T) \subseteq \bigcap\{\sigma(T+J) : J \in \mathscr{F}(A)\}$ follows from the formula above. Now suppose $\lambda \notin \sigma_W(T)$, so $\lambda - T \in \Phi^0(X)$. Then by Theorem 9, $\exists J \in \mathscr{F}(A)$ such that $(\lambda - T) - J$ is invertible. Thus, $\lambda \notin \sigma(T+J)$. This establishes the reverse inclusion.

4. Characterization

In this section we give a characterization in terms of spectral properties of T for when there exists an EUN or an ELN in $\mathrm{Lat}(T)$.

For $\lambda \in \mathbf{C}$ and m a positive integer, let $N(\lambda, m) = \mathscr{N}((\lambda - T)^m)$. Any vector in $N(\lambda, m)$ for some λ and m is called a principal vector of T, and we let $\mathscr{P}(T)$ denote the set of all principal vectors of T.

Theorem 10. *Let X be a separable Banach space, and let $T \in B(X)$.*

(1) $\exists \mathscr{E}$ an EUN with $\mathscr{E} \subseteq \mathrm{Lat}(T)$ if and only if X is the closed linear span of $\mathscr{P}(T)$.

(2) $\exists \mathscr{E}$ an ELN with $\mathscr{E} \subseteq \mathrm{Lat}(T)$ if and only if the closed linear span of $\mathscr{P}(T^*)$ is a separable total subspace of X^*.

Proof. First assume $\mathscr{E}$ is an EUN with $\mathscr{E} = \{M_n : 0 \leq n \leq \infty\} \subseteq \mathrm{Lat}(T)$. Now each M_n is f.d. and $T(M_n) \subseteq M_n$. By [6, pp. 336–338], M_n is the span of principal vectors of T. But the closed linear span of $\{M_n : n \geq 0\}$ is all of X, so the closed linear span of $\mathscr{P}(T)$ is X.

Conversely, assume X is the closed linear span of $\mathscr{P}(T)$. For $x \in N(\lambda, m)$, let $M(x) = \mathrm{span}\{x, Tx, T^2x, \dots, T^{m-1}x\}$. We claim that $M(x)$ is T-invariant. To prove this it suffices to show that $T^m x \in M(x)$. This follows from the equation $0 = (\lambda - T)^m x = \sum_{k=0}^{m} \binom{n}{k} \lambda^k (-T)^{m-k} x$. Let $X_0 = \mathrm{span}(\mathscr{P}(T))$. Let $\{y_n\}_{n \geq 1}$ be a countable dense subset of X_0. Each y_n has the form

$$y_n = \sum_{k=1}^{m_n} \lambda_k x_{n,k}$$

where each $x_{n,k}$ is a principal vector of T. Then

$$W = \{M(x_{n,k}) : 1 \leq k \leq m_n,\ n \geq 1\}$$

is a collection of T-invariant f.d. subspaces of X, and as $\{y_n\}_{n \geq 1} \subseteq \mathrm{span}(W)$, we have $\mathrm{span}(W)^- = X$. Relabel the subspaces in W as $\{Y_n : n \geq 1\}$. Set $M_0 = \{0\}$, $M_n = \mathrm{span}\{Y_k : 1 \leq k \leq n\}$, $M_\infty = X$. Then $\mathscr{E} = \{M_n : 0 \leq n \leq \infty\}$ is an EUN and $\mathscr{E} \subseteq \mathrm{Lat}(T)$.

Assume $\mathscr{E} = \{M_n : 0 \leq n \leq \infty\}$ is an ELN with $\mathscr{E} \subseteq \mathrm{Lat}(T)$. Since M_n has finite codimension in X, it follows that $M_n^\perp$ is f.d. for $1 \leq n < \infty$. Also, $M_n^\perp$ is T^*-invariant for all n. Now the closed linear span of $\{M_n^\perp : 1 \leq n < \infty\}$ is a separable and total subspace of X^*. The argument in the proof of (1) shows that $\mathrm{span}(\mathscr{P}(T^*))^-$ contains this subspace.

Conversely, assume $\mathrm{span}(\mathscr{P}(T^*))^-$ is separable and total in X^*. As in the argument for part (1), $\exists \{W_n : n \geq 0\}$ such that each W_n is a f.d., T^*-invariant subspace, $W_0 = \{0\}$, $W_n \subseteq W_{n+1}$, and

$$\mathrm{span}(\mathscr{P}(T^*))^- = \mathrm{span}\{W_n : n \geq 1\}^-.$$

Let $M_n = \{x \in X : \alpha(x) = 0 \text{ for all } \alpha \in W_n\}$ for $n \geq 0$. Set $M_0 = X$, $M_\infty = \{0\}$. It is easy to check that $\mathscr{E} = \{M_n : 0 \leq n \leq \infty\}$ is an ELN with $\mathscr{E} \subseteq \mathrm{Lat}(T)$.

5. Open problems

When $\mathscr{E}$ is either an EUN or an ELN, and $\mathscr{E} \subseteq \mathrm{Lat}(T)$, then this fact affects the spectral and Fredholm properties of T. We believe that a similar situation holds in the case of certain Volterra-type integral operators.

Let X be some Banach space of measurable functions on $[0, \infty)$, $L^p[0, \infty)$ for example. Assume that $K(x, t)$ is a measurable function such that the integral operator

$$V(f)(x) = \int_0^x K(x, t) f(t)\, dt \qquad (f \in X)$$

is a bounded operator on X. For $a \geq 0$, let

$$M_a = \{f \in X : f \equiv 0 \text{ a.e. on } [0, a]\}, \qquad M_\infty = \{0\}.$$

Then $\mathfrak{M} = \{M_a : 0 \leq a \leq \infty\}$ is a continuous nest and $\mathfrak{M} \subseteq \mathrm{Lat}(V)$. Two open questions concerning V are:

Question 1. Is $\sigma(V)$ connected?

Question 2. If $\lambda - V \in \Phi(X)$, then is $\mathrm{ind}(\lambda - V) \leq 0$?

These questions are considered in Barnes' paper [1].

There are similar open questions concerning the operator

$$U(f)(x) = \int_x^\infty K(x, t) f(t)\, dt \qquad (f \in X),$$

assuming that $U \in B(X)$. Note that $\mathrm{Lat}(U)$ contains the continuous nest $\mathfrak{N} = \{N_a : 0 \leq a \leq \infty\}$, where

$$N_a = \{f \in X : f \equiv 0 \text{ a.e. on } [a, \infty)\}, \qquad N_\infty = X.$$

Question 3. Is $\sigma(U)$ connected?

Question 4. If $\lambda - U \in \Phi(X)$, then is $\mathrm{ind}(\lambda - U) \geq 0$?

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